

Padé Estimate of QCD's Infrared Boundary

F.A. Chishtie*, V. Elias†
Department of Applied Mathematics
University of Western Ontario
London, Ontario, N6A 3K7 Canada

T.G. Steele‡
Department of Physics and Engineering Physics
University of Saskatchewan
Saskatoon, Saskatchewan, S7N 5E2
Canada

February 1, 2008

Abstract

A mass scale μ_c representing the boundary between effective theories of strong interactions and perturbative QCD (infrared and ultraviolet regimes) exists when the β -function is characterized by a simple pole. This behaviour, which is known to occur in $N = 1$ supersymmetric Yang-Mills theories, leads to an infrared attractor in the evolution of the coupling constant, with the mass scale μ_c of the attractor providing a natural boundary between the infrared and ultraviolet regimes. It is demonstrated that [2|2], [3|1] and [1|3] Padé-approximant versions of the three-flavour ($n_f = 3$) QCD β -function each contain a simple pole corresponding to such an infrared attractor. All three approximants, separately considered, are seen to lead to nearly equivalent estimates for the mass scale μ_c .

Although QCD is well-understood both qualitatively and quantitatively as a perturbative theory for the strong interactions, we have a surprisingly limited amount of information about the infrared boundary of its perturbative domain. Such a boundary (which is also anticipated by arguments in which hadrons and quarks are dual field variables for weak and strong QCD [1]) must exist to separate effective strong interaction theories (*e.g.*, chiral perturbation theory, linear sigma model, *etc.*) from the higher-momentum region characterised by the perturbative quantum field theory of quarks and gluons. Consequently, one can argue that the infrared behaviour of the perturbative QCD coupling should *not* be characterised by smooth evolution to an infrared-stable fixed point, but rather by behaviour that would clearly separate the infrared domain of effective strong-interaction theories from perturbative QCD. Indeed, there exists both lattice [2], analytical [3], and Padé-approximant [4, 5] corroboration for the idea that an infrared-stable fixed point does not occur within QCD unless the number of active fermion flavours contributing to the evolution of the QCD couplant is substantially larger than three.

A clear demarcation between the infrared and ultraviolet regions would occur if the $n_f = 3$ QCD β -function were characterised not by a positive zero corresponding to an infrared-stable fixed point, but rather by a positive *pole*, an infrared-attractive singularity in the β -function occurring at a momentum scale which necessarily corresponds to a lower bound on the domain of the running QCD coupling constant (assumed here to be real). Such behaviour, schematically presented in Fig. 1, is known to characterise the exact β -function for $N = 1$ supersymmetric Yang-Mills theory in the NSVZ renormalisation scheme [6], a “supersymmetric gluodynamics” whose β -function pole forms an infrared-attractive terminal point for coupling-constant evolution within the theory’s asymptotically-free

*email: fachtisht@uwo.ca

†email: velias@uwo.ca

‡email: Tom.Steele@usask.ca

phase [7]. There exists evidence from Padé-approximant methods that similar dynamics may also characterise the infrared region of QCD [4]. In the present note, we discuss whether such methods can provide information as to the actual mass scale which separates the infrared and ultraviolet domains of strong interaction physics, as well as the strong-interaction couplant magnitude characterising this mass scale.

Weighted asymptotic Padé-approximant (WAPAP) methods have been utilised to estimate the n_f -dependence of the five-loop contribution (β_4) to the QCD β -function, defined here for $x(\mu) \equiv \alpha_s(\mu)/\pi$ to be

$$\mu^2 \frac{dx}{d\mu^2} = - \sum_{k=0}^{\infty} \beta_k x^{k+2} = -\beta_0 x^2 (1 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4 + \dots) \quad . \quad (1)$$

For $n_f = 3$, the known $\overline{\text{MS}}$ β -function coefficients in (1) are [8]

$$\beta_0 = \frac{9}{4} \quad , \quad \beta_1 = 4 \quad , \quad \beta_2 = \frac{3863}{384} \quad , \quad \beta_3 = 47.2280 \quad . \quad (2)$$

The WAPAP estimate of the n_f -dependence of the five-loop term (inclusive of quadratic-Casimir contributions to β_3) is found to be [9]

$$\beta_4 \cong \frac{1}{45} (7.59 \times 10^5 - 2.19 \times 10^5 n_f + 2.05 \times 10^4 n_f^2 - 49.8 n_f^3 - 1.84 n_f^4) \xrightarrow{n_f=3} 278 \quad , \quad (3)$$

$$R_4 = \frac{\beta_4}{\beta_0} \xrightarrow{n_f=3} 124 \quad . \quad (4)$$

This estimate may be used to construct non-trivial $N + M = 4$ $[N|M]$ -Padé approximants that reproduce the known coefficients R_1 – R_3 and the estimate (4) for R_4 within the β -function series of (1):

$$\beta^{[2|2]}(x) = -\frac{9}{4} x^2 \frac{[1 - 5.4983x - 1.9720x^2]}{[1 - 7.2762x + 6.4923x^2]} \quad (5)$$

$$\beta^{[1|3]}(x) = -\frac{9}{4} x^2 \frac{[1 - 5.7397x]}{[1 - 7.5174x + 8.8933x^2 - 3.1896x^3]} \quad (6)$$

$$\beta^{[3|1]}(x) = -\frac{9}{4} x^2 \frac{[1 - 4.1155x - 6.0058x^2 - 5.3589x^3]}{[1 - 5.8932x]} \quad (7)$$

All three approximants above exhibit a positive pole that precedes any positive zeroes, consistent with infrared dynamics analogous to those of NSVZ supersymmetric gluodynamics, as discussed above.¹ This first positive pole, which corresponds to the couplant magnitude at the infrared-boundary momentum scale μ_c , is surprisingly comparable for all three approximants:

$$\begin{aligned} [2|2]: \quad x(\mu_c) &= 0.160 \quad , \\ [1|3]: \quad x(\mu_c) &= 0.162 \quad , \\ [3|1]: \quad x(\mu_c) &= 0.170 \quad . \end{aligned} \quad (8)$$

The fact that *all three* approximants exhibit a positive pole of nearly equivalent magnitude (which precedes any positive zeroes) is indicative that such a pole is not likely to be an artefact defect pole [10], but rather a manifestation of a true pole within the underlying all-orders β -function. It is to be noted that the pole couplant magnitude, as estimated in (8), is itself sufficiently small for perturbative physics to remain viable near the infrared boundary. This behaviour is very different from an infrared-slavery scenario in which the QCD couplant grows non-perturbatively large in the vicinity of a deep-infrared Landau singularity.

Similar consistency of poles obtained from differing Padé approximants to the perturbative β -function has already been shown to characterise supersymmetric gluodynamics in both NSVZ and in DRED renormalization

¹The fact that a positive pole is always found to precede any positive zeroes for all three approximants has already been established [4] for *arbitrary* values of R_4 .

schemes [11]. We reiterate that the pole in the former of these two schemes is known to occur from the all-orders β -function expression, whether derived via NSVZ instanton calculus [6] or via imposition of the Adler-Bardeen theorem upon the supermultiplet structure of the theory [11, 12].

One can utilise the infrared-attractive couplant values (8) in order to obtain separate estimates of the infrared boundary μ_c for each approximant considered. For specific $n_f = 3$ $[N|M]$ approximant versions of the QCD β -function, one finds that

$$\mu_c^{[N|M]} = m_\tau \exp \left[\frac{1}{2} \int_{x(m_\tau)}^{x(\mu_c)} \frac{dx}{\beta^{[N|M]}(x)} \right]. \quad (9)$$

We utilise the approximants (5)–(7) within the integrand of (9), as well as the respective values (8) for $[2|2]$, $[1|3]$, and $[3|1]$ approximant values of $x(\mu_c)$ for the upper bound of integration. For the lower bound of integration, we assume $\alpha_s^{MS}(m_\tau) = \pi x(m_\tau) = 0.33 \pm 0.02$, consistent with recent analyses [13]. We then obtain via (9) the following values for the infrared-boundary momentum scale:

$$\begin{aligned} \mu_c^{[2|2]} &= 1.14 \pm 0.11 \text{ GeV} \\ \mu_c^{[1|3]} &= 1.13 \pm 0.11 \text{ GeV} \\ \mu_c^{[3|1]} &= 1.09 \pm 0.11 \text{ GeV} \end{aligned} \quad (10)$$

These results not only exhibit remarkable consistency with each other, but also support the identification of perturbative QCD's infrared boundary with a momentum scale at (or somewhat above) the mass scale characterising nucleons and the vector meson octet. This picture is quite different from the usual one of an $\mathcal{O}(300 \text{ MeV})$ value for the Landau singularity Λ_{QCD} characterising coupling constant evolution via the *truncated* β -function series. We reiterate that the estimates (8) and (10) for the infrared terminus of the couplant evolution within the asymptotically free phase of $n_f = 3$ QCD are consistent with both the applicability of controllably-perturbative QCD down to $\mathcal{O}(1 \text{ GeV})$ momentum scales, as well as the necessity for alternative descriptions (*e.g.* effective Lagrangians and hadronic field variables) to characterise sub-GeV (or sub- $4\pi f_\pi$ [15]) strong-interaction physics.

The results (9) and (10) are, of course, sensitive to input information. An across-the-board 150 MeV decrease from (10) in the estimated range of μ_c is seen to occur if we choose the three-flavour threshold at $\mu_t = m_c(m_c) \cong 1.25 \text{ GeV}$ [14]. More significantly, the set of values (10) for μ_c relies ultimately on ref. [9]'s WAPAP estimate (3,4) for the five-loop contribution to the QCD β -function. The $n_f = 3$ estimate obtained via (3) involves relatively small differences of large numbers. Consequently, the individual coefficients of powers of n_f are likely to be more accurate than the estimate of β_4 and R_4 obtained by evaluating the full polynomial in n_f .²

To test the uniformity of the infrared-boundary mass scale obtained via different approximants, let us choose R_4 for each approximant so as to ensure the occurrence of a positive pole at $\alpha_s(\mu_c) = \pi/4$ (*i.e.*, at $x = 1/4$). This particular choice is motivated both as the critical value of the strong coupling for chiral symmetry breaking [17] as well as by the couplant value characterising Nambu-Jona-Lasinio and linear-sigma-model approaches toward the generation of a dynamical quark mass [18].

The appropriate set of approximants for the $n_f = 3$ β -function series

$$\mu^2 \frac{dx}{d\mu^2} = -\frac{9}{4}x^2 \left[1 + \frac{16}{9}x + \frac{3863}{864}x^2 + 20.9902x^3 + R_4x^4 + \dots \right] \quad (11)$$

² Such is the case for a prior asymptotic Padé-approximant prediction of the four-loop term [16], $\beta_3^{pred} = \{23600 - 6400n_f + 350n_f^2 + 1.499n_f^3\}/256$, whose polynomial coefficients are in good term-by-term agreement with those of the subsequent exact calculation [8], $\beta_3^{true} = \{29243.0 - 6946.30n_f + 405.089n_f^2 + 1.499n_f^3\}/256$. When $n_f = 3$, however, the estimated value $\beta_3^{pred} = 29.65$ differs from $\beta_3^{true} = 47.228$ by a relative-error (37%) whose magnitude is a factor of two or more larger than that of the relative error characterising each estimated polynomial coefficient (19%, 7.9%, and 14%, respectively).

	$\alpha_s(m_\tau) = 0.31$	$\alpha_s(m_\tau) = 0.33$	$\alpha_s(m_\tau) = 0.35$
$\mu_c^{[2 2]}$	785 MeV	874 MeV	960 MeV
$\mu_c^{[1 3]}$	778 MeV	867 MeV	952 MeV
$\mu_c^{[3 1]}$	775 MeV	863 MeV	948 MeV

Table 1: Values of $\mu_c^{[N|M]}$, as obtained via (9), for $[N|M]$ approximants to the $n_f = 3$ QCD β -function. The (unknown) five-loop contribution R_4 to the perturbative series (11) is chosen (see (15)) to ensure a pole at $x(\mu_c) = 1/4$ for all three Padé-approximant cases.

with R_4 arbitrary is

$$\beta^{[2|2]}(x) = -\frac{9}{4}x^2 \frac{[1 + (7.19456 - 0.102610R_4)x + (-11.3292 + 0.0756438R_4)x^2]}{[1 + (5.41678 - 0.102610R_4)x + (-25.4301 + 0.258062R_4)x^2]}, \quad (12)$$

$$\beta^{[1|3]}(x) = -\frac{9}{4}x^2 \frac{[1 + (5.80845 - 0.0933552R_4)x]}{[1 + (4.03067 - 0.0933552R_4)x + (-11.6367 + 0.165965R_4)x^2 + (-18.3242 + 0.122349R_4)x^3]}, \quad (13)$$

$$\beta^{[3|1]}(x) = -\frac{9}{4}x^2 \frac{[1 + (1.77778 - 0.0476412R_4)x + (4.447106 - 0.0846954R_4)x^2 + (20.9902 - 0.213007R_4)x^3]}{[1 - 0.0476412R_4x]}. \quad (14)$$

The first positive pole for all three approximants is seen to precede any positive zeroes and is clearly dependent on the value of R_4 . One can easily show that all three approximants acquire a positive pole at $x = 1/4$ for very similar values of R_4 :

$$\begin{aligned} [2|2]: \quad R_4 &= 80.307 \\ [1|3]: \quad R_4 &= 89.925 \\ [3|1]: \quad R_4 &= 83.961 \end{aligned} \quad (15)$$

Given $\alpha_s(m_\tau)$ and $x(\mu_c) = 1/4$, as ensured by these respective choices for R_4 , one can utilise (9) to estimate corresponding mass scales μ_c for the three approximants. In Table 1, such values are obtained via couplant evolution from three different choices for $\alpha_s(m_\tau)$ in the range $\alpha_s(m_\tau) = 0.33 \pm 0.02$ [13]. In all these estimates, three-flavour coupling constant evolution is assumed to be valid below m_τ .

Table 1 shows striking uniformity in the values for the infrared boundary mass scale μ_c obtained via three distinct Padé-approximants. Moreover, the three approximants generate values for μ_c very near the ρ -meson mass when the lowest value for $\alpha_s(m_\tau)$ is chosen.

Indeed, Figure 2 demonstrates that the infrared boundary for all three approximants is nearly equivalent *even if R_4 is allowed to vary arbitrarily*. The corresponding mass scale μ_c , as plotted in Figure 2, is obtained for each approximant by substituting (12–14) into the integrand of (9), with (9)’s upper bound of integration identified with the first positive poles of (12–14).³ For a given choice of R_4 , Figure 2 shows that the infrared-boundary mass scales characterising all three approximants are surprisingly close. The $\mathcal{O}(600 \text{ MeV})$ lower bound evident from the figure for all three approximants is particularly striking. Different Padé approximants to the $n_f = 3$ QCD β -function thus appear to be quite consistent in the infrared dynamics they predict, suggestive that similar pole-driven dynamics may characterise the infrared boundary of QCD itself.

The above results have all been obtained in the $\overline{\text{MS}}$ scheme, an explicitly gauge-invariant renormalization procedure for which the β -function has been explicitly calculated to four-loop order, thereby facilitating our use of

³When R_4 becomes negative, the $[3|1]$ approximant version of the $n_f = 3$ β -function no longer exhibits a positive pole (or any positive zeroes, as would be the case with an infrared stable fixed point), but still exhibits a Landau singularity extractable from (9) provided the upper bound of integration in (9) is replaced by infinity. The $[3|1]$ approximant curve in Figure 2 displays the mass scale for this Landau singularity when R_4 is negative. The figure shows this mass scale to be consistent with the mass scales characterising the simple poles within the $[2|2]$ - and $[1|3]$ -approximant versions of the β -function for the same negative values of R_4 .

Padé approximants to the series (1). Moreover, perturbative QCD phenomenology is more readily available (and more likely to be corroborated) in $\overline{\text{MS}}$ than in other renormalization schemes. The question remains, however, as to whether the infrared boundary we observe is a peculiarity of the $\overline{\text{MS}}$ scheme we employ. In eq. (1), the coefficients R_2, R_3, R_4, \dots , (corresponding to β_k with $k \geq 2$) are all scheme-dependent, and therefore negotiable within the context of formal perturbative QCD. Indeed, a QCD renormalization scheme proposed by 't Hooft [19] in which $\beta_k = 0$ for $k \geq 2$ is guaranteed *by construction* to be free of β -function poles.⁴

Recent work [20] has developed a procedure by which the leading renormalization-scheme dependence (*i.e.*, explicit dependence on R_2) can be eliminated from a perturbative QCD result. This approach leads ultimately to β -functions characterized by the remaining renormalization-scheme parameters (*i.e.*, R_3, \dots). A reasonable set of Padé approximants to β -functions extracted by this procedure is shown in [20] to exhibit poles of a magnitude ($0.29 \lesssim x \lesssim 0.33$) sufficiently large for chiral symmetry breaking at the infrared boundary of QCD.

Acknowledgments: We are grateful for useful discussions with V.A. Miransky, and for support from the Natural Sciences and Engineering Research Council of Canada (NSERC).

References

- [1] N. Seiberg and E. Witten, Nucl. Phys. B 426 (1994) 19 and B 431 (1994) 484.
- [2] Y. Iwasaki, K. Kanaya, S. Sakai, T. Yoshié, Phys. Rev. Lett. 69 (1992) 21.
- [3] T. Banks and A. Zaks, Nucl. Phys. B 196 (1982) 189;
T. Appelquist, J. Terning and L.C.R. Wijewardhana, Phys. Rev. Lett. 77 (1996) 1214;
V.A. Miransky and K. Yamawaki, Phys. Rev. D 55 (1997) 5051 and Erratum 56 (1997) 3768;
M. Velkovsky and E. Shuryak, Phys. Lett. B437 (1998) 398;
T. Appelquist, A. Ratnaweera, J. Terning and L.C.R. Wijewardhana, Phys. Rev. D58 (1998) 105017;
E. Gardi and M. Karliner, Nucl. Phys. B529 (1998) 383;
E. Gardi, G. Grunberg and M. Karliner, JHEP 07 (1998) 007;
V. A. Miransky, Phys. Rev. D59 (1999) 105003;
E. Gardi and G. Grunberg, JHEP 9903 (1999) 024.
- [4] F.A. Chishtie, V. Elias, V.A. Miransky, T.G. Steele, Prog. Theor. Phys. 104 (2000) 603.
- [5] V. Elias, T.G. Steele, F. Chishtie, R. Migneron and K. Sprague, Phys. Rev. D 58 (1998) 116007;
F.N. Ndili, hep-ph/0011116 (to appear Phys. Rev. D).
- [6] V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, Nucl. Phys. B229 (1983) 381.
- [7] I.I. Kogan and M. Shifman, Phys. Rev. Lett. 75 (1995) 2085.
- [8] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, Phys. Lett. B400 (1997) 379.
- [9] J. Ellis, I. Jack, D.R.T. Jones, M. Karliner and M.A. Samuel, Phys. Rev. D 57 (1998) 2665.
- [10] G. Baker, P. Graves-Morris, *Padé Approximants* [Vol. 13 of *Encyclopedia of Mathematics and its Applications*] (Addison-Wesley, Reading, MA, 1981) pp. 48–57.
- [11] V. Elias, J. Phys. G 27 (2001) 217.
- [12] D.R.T. Jones, Phys. Lett. B 123 (1983) 45.
- [13] ALEPH Collaboration (R. Barate et al.), Eur. Phys. J. C 4 (1998) 409;
G. Cvetič and T. Lee, hep-ph/0101297;
C.J. Maxwell and A. Mirjalili, hep-ph/0103164.

⁴Curiously, the existence and approximate size of the n_f threshold for β -function zeroes (*i.e.*, infrared-stable fixed points) in the 't Hooft scheme is corroborated by Padé-approximants constructed from the known terms of the $\overline{\text{MS}}$ β -function [4].

- [14] Particle Data Group (D.E. Groom *et al*), Eur. Phys. J. C 15 (2000) 1.
- [15] A. Manohar and H. Georgi, Nucl. Phys. B 234 (1984) 189.
- [16] J. Ellis, M. Karliner, and M. A. Samuel, Phys. Lett. B 400 (1997) 176.
- [17] P. Fomin and V.A. Miransky, Phys. Lett. B 64 (1976) 166;
P.I. Fomin, V.P. Gusynin, V.A. Miransky, and Yu. A. Sitenko, Riv. Nuovo Cim. 6 (1983) 1;
K. Higashijima, Phys. Rev. D 29 (1984) 1228.
- [18] V. Elias and M. D. Scadron, Phys. Rev. Lett. 53 (1984) 1129;
L. R. Baboukhadia, V. Elias, and M. D. Scadron, J. Phys. G 23 (1997) 1065.
- [19] G. 't Hooft, *Recent Developments in Gauge Theories*, [Vol. 59 of NATO Advanced Study Institute Series B: Physics, ed. G. 't Hooft *et al.*], (Plenum New York, 1980).
- [20] G. Cvetič, Phys. Lett. B 486 (2000) 100.

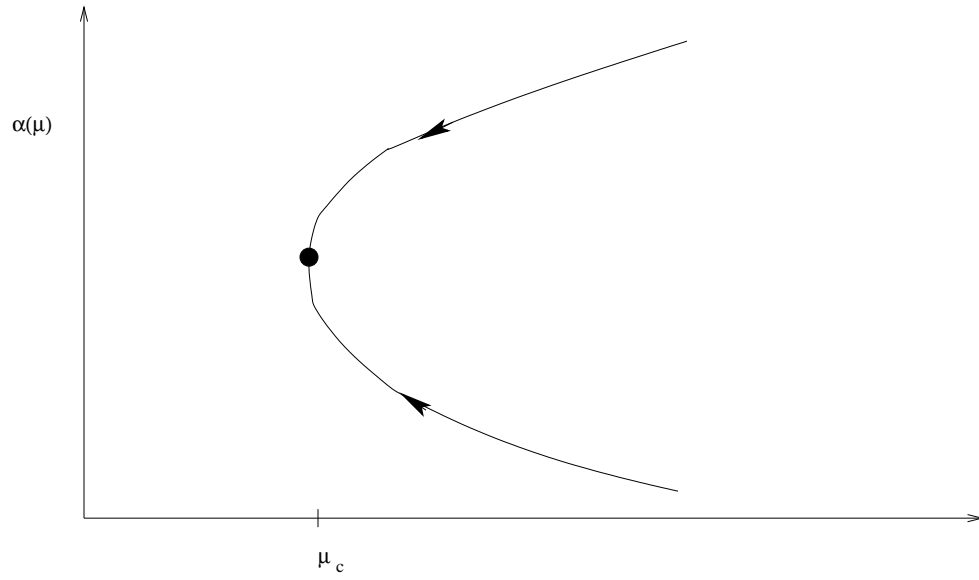


Figure 1: Qualitative behaviour of the running coupling $x(\mu)$ in an asymptotically-free theory with an infrared attractor devolving from a simple pole in the β -function.

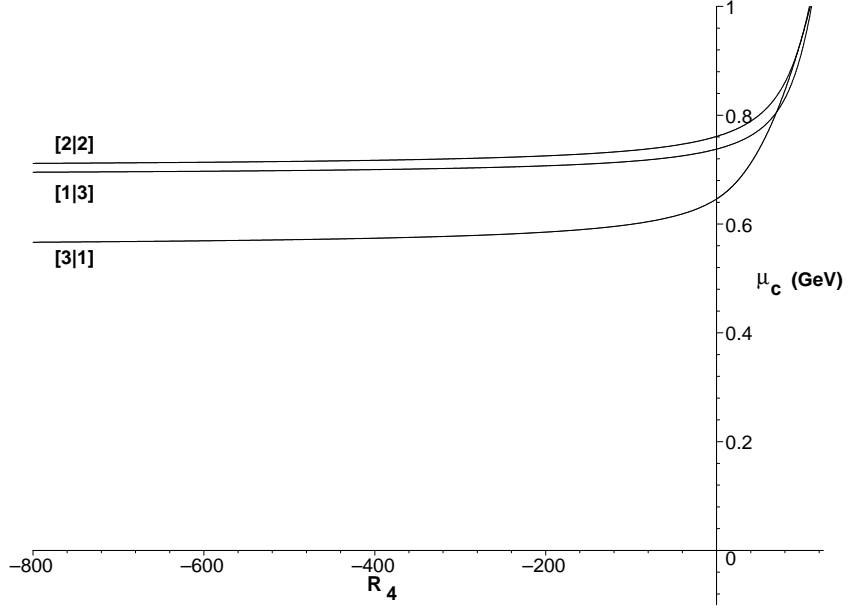


Figure 2: Dependence of the infrared-boundary scale μ_c on the five-loop β -function coefficient $R_4 = \beta_4/\beta_0$, based upon $n_f = 3$ evolution of the couplant from an initial value of $\alpha_s(m_\tau) = 0.33$. We only plot values of R_4 less than the WAPAP estimate (4); if R_4 is allowed to increase much past this value, the first positive poles of (12–14) are soon seen to be reached at values of μ exceeding the four-flavour threshold μ_t .